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PROPAGATION OF REVERSIBLE DEFORMATION IN A MEDIUM WITH ACCUMULATED IRREVERSIBLE STRAINS

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The paper studies the effect of previously accumulated plastic strains on the conditions of existence and propagation of the discontinuity surfaces of elastic strains arising from subsequent shock loading. It is shown that the anisotropy of properties of the medium due to the presence of previous irreversible strains influences the propagation of repeated boundary perturbations in an elastoplastic medium. Plane shock waves propagating in a one-dimensional, uniformly deformed medium are considered in detail. Solutions are given of the problem of instantaneous loading and unloading of an elastoplastic half-space with irreversible strains.

Introduction. The existence of a free state is one of the main hypotheses used in modeling the deformation of materials. In this case, the strains are reckoned from a certain state in which they are set equal to zero. Stresses in the free state are also considered zero. Because of the various heterogeneities intrinsic to real materials, the free state in them practically does not occur. In the processes of pretreatment and manufacture of articles (rolling, forming), materials can acquire considerable irreversible strains. Such strains can be caused by residual stresses, which are not completely eliminated by annealing and quenching. Therefore, the accumulated irreversible strains can influence subsequent deformation, in particular, the nature of distribution of boundary perturbations on materials thus deformed. In the present paper, we study such effects in a body with accumulated plastic strains under shock loading.

Strong discontinuity surfaces in elastoplastic bodies were studied in [1–3]. However in those studies, strains were considered small and irreversible strain discontinuity surfaces were studied. In the present paper, we focus on the effect of previous irreversible strains on the conditions of existence and propagation of reversible strain discontinuity surfaces. Obviously, such formulation is meaningless for small strains, and, hence, in the present paper, we use the model of an elastoplastic medium with finite strains, both irreversible and reversible.

1. Basic Model Relations. Finite strain theory is based on the method of separation of experimental total strains into reversible (elastic) and irreversible (plastic) components, which cannot be measured experimentally. Existing models of large elastoplastic strains [4–9] differ mainly in the method of such separation. More often, these models use the assumption that any real state of a deformed body corresponds to another unique state called an unloaded state [4, 6, 9]. In this connection, it becomes necessary to choose [6] an objective derivative that relates the plastic strain tensor to the irreversible strain rate tensor. In [7, 8], the elastic and plastic strain tensors are defined by differential relations (transfer equations). Without going into detail, which are described in [7, 8, 10], we shall indicate the relations necessary for further consideration. In separating total strains into reversible and irreversible components, we assume that plastic strains do not vary during unloading, and, hence, the components of this tensor vary similarly to those in rigid-body rotation, i.e., $p_{ij} = z_{ki}p^0_{km}z_{mj}$. Here z_{ij} are the orthogonal tensor components, which, generally speaking, can be different at each point of the medium and are determined by the variable elastic strains e_{ij} and the strain rate ε_{ij}. As p^0_{km}, we can adopt values of the components p_{km} at the beginning of unloading, which is equivalent to the differential relation

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$$\frac{dp_{ij}}{dt} = r_{ik}p_{kj} - p_{ik}r_{kj}, \qquad r_{ij} = -r_{ij} = \frac{v_{i,j} - v_{j,i}}{2} + M_{ij}(e_{ij}, \varepsilon_{ij}),$$
$$v_i = \frac{du_i}{dt} = \frac{\partial u_i}{\partial t} + v_j u_{i,j}, \qquad \varepsilon_{ij} = \frac{v_{i,j} + v_{j,i}}{2}, \tag{1.1}$$

where u_i are the particle displacement vector components, the skew-symmetric tensor M_{ij} determines the nonlinear effect of elastic deformation during unloading on the variation of the plastic strain tensor components (an explicit relation for it is given in [7]); the Latin subscript after comma denotes partial differentiation with respect to a corresponding spatial coordinate ($v_{i,j} = \partial v_i/\partial x_j$); the summation is performed over repeated subscripts. For the regions of active loading, according to the laws of thermodynamics, relations (1.1) must be supplemented by the following equations of variation (transfer) [8]) of the elastic (e_{ij}) and plastic (p_{ij}) strain tensors:

$$\frac{de_{ij}}{dt} = \varepsilon_{ij} - \varepsilon^p_{ij} - \frac{1}{2}(e_{ik}v_{k,j} + v_{k,i}e_{kj} - r_{ik}e_{kj} + e_{ik}r_{kj} - \varepsilon^p_{ik}e_{kj} + e_{ik}\varepsilon^p_{kj}),$$
$$\frac{dp_{ij}}{dt} = \varepsilon^p_{ij} - \varepsilon^p_{ik}p_{kj} - p_{ik}\varepsilon^p_{kj} + r_{ik}p_{kj} - p_{ik}r_{kj}. \tag{1.2}$$

From this, for the metric tensor components $g_{ij} = a_{k,i}a_{k,j}$ (a_k are material coordinates of points of the medium) and Almansi strain tensor $d_{ij} = 0.5(\delta_{ij} - g_{ij})$), we obtain the algebraic representations

$$g_{ij} = (\delta_{ik} - e_{ik})(\delta_{km} - 2p_{km})(\delta_{mj} - e_{mj}), \quad d_{ij} = e_{ij} + p_{ij} - e_{ik}e_{kj}/2 - e_{ik}p_{kj} - p_{ik}e_{kj} + e_{ik}p_{ks}e_{sj}. \tag{1.3}$$

For unloading ($\varepsilon^p_{ij} = 0$), the second equality in (1.2) directly leads to (1.1), i.e., invariance of the irreversible strain tensor p_{ij}. Thus, the differential relations (1.2), together with (1.3), can be treated as a definition of the tensors e_{ij}, p_{ij}, and ε^p_{ij}. In this case, the second equality in (1.2) represents the objective derivative that relates the plastic strain tensor p_{ij} to the plastic strain rate tensor ε^p_{ij}.

Along with (1.2), a consequence of the laws of thermodynamics is the following analog of the Murnagan formula known in the nonlinear theory of elasticity:

$$\sigma_{ij} = \frac{\rho}{\rho_0}\frac{\partial W}{\partial e_{ik}}(\delta_{kj} - e_{kj}). \tag{1.4}$$

Here ρ and ρ_0 are the densities of the medium in the current and undeformed states and σ_{ij} are the components of the Euler–Cauchy stress tensor. In the derivation of (1.4) in [7], it was assumed that the free-energy density $\psi = \rho_0^{-1}W$ is a function of only reversible strains e_{ij} and does not depend on the irreversible strains p_{ij}. The latter determine the dissipation mechanism of an elastoplastic medium. The elastic properties of an isotropic medium can be determined by a standard method using a power series expansion of the elastic potential $W = W(I_1, I_2, I_3)$ in the state with zero elastic strains:

$$W = \lambda I_1^2/2 + \mu I_2 + lI_1I_2 + mI_1^3 + nI_3 + \ldots,$$
$$I_1 = e_{jj} - e_{jk}e_{kj}/2, \quad I_2 = e_{ij}e_{ji} - e_{ij}e_{jk}e_{ki}, \quad I_3 = e_{ik}e_{ks}e_{si}. \tag{1.5}$$

The invariants of the reversible strain tensor in (1.5) are chosen such that as plastic strains tend to zero, relation (1.5) becomes the well-known representations of elasticity nonlinear theory:

$$W = \lambda J_1^2/2 + \mu J_2 + lJ_1J_2 + mJ_1^3 + nJ_3 + \ldots, \quad J_1 = d_{kk},\ J_2 = d_{ij}d_{ji},\ J_3 = d_{ij}d_{jk}d_{ki},$$
$$\sigma_{ij} = \frac{\rho}{\rho_0}\frac{\partial W}{\partial d_{ik}}(\delta_{kj} - 2d_{kj}). \tag{1.6}$$

Below, plastic flow is considered ideal, and, therefore, we postulate the existence of a fixed loading surface $f(\sigma_{ij}, p_{ij}) = k$ and adopt the conditions of the Mises maximum principle, whose consequence for active loading is the associate plastic flow rule.

2. Relations on a Discontinuity Surface. Let a discontinuity surface of $\Sigma(t)$ reversible strains propagate in an elastoplastic medium. The position of the surface at any time is defined by

$$x_i = x_i(y^1, y^2, t). \tag{2.1}$$

The surface coordinates y^β ($\beta = 1, 2$) in (2.1) are assumed to be orthogonal. At each point on the surface $\Sigma(t)$, where a single normal with components ν_i is defined ($\nu_i\nu_i = 1$), we have $\nu_i x_{i,\beta} = 0$ [11]. The discontinuities of the components of the displacement tensor gradient are written as

$$[u_{i,j}] = \tau\nu_i\nu_j + g^{\alpha\beta}\tau_\alpha x_{i,\beta}\nu_j = \tau\nu_i\nu_j + \tau^\beta x_{i,\beta}\nu_j = (\tau\nu_i + \gamma\mu_i)\nu_j,$$
$$\tau = [u_{k,k}], \quad \gamma^2 = \tau^\beta\tau_\beta, \quad \mu_i = \tau^\beta\gamma^{-1}x_{i,\beta}, \quad \mu_i\mu_i = 1, \quad \mu_i\nu_i = 0. \tag{2.2}$$

Here $g^{\alpha\beta}$ are the contravariant components of the surface metric tensor, such that $g^{\alpha\beta}g_{\beta\gamma} = \delta^\alpha_\gamma$ and $g_{\beta\gamma} = x_{i,\beta}x_{i,\gamma}$. Unit vectors with the components ν_i and μ_i define the polarization plane of the shock wave $\Sigma(t)$ at the time considered. The quantities τ and γ should be called the longitudinal and transverse intensities of the discontinuity on $\Sigma(t)$. The discontinuity of the quantity in (2.2) is denoted by square brackets. For definiteness, we assume that $[a] = a^+ - a^-$ [a^+ and a^- are values of a discontinuous quantity on $\Sigma(t)$, calculated ahead of and directly behind the surface $\Sigma(t)$]. Representation (2.2) implies the following relations for the discontinuities of the total strains d_{ij} and the particle velocities v_i:

$$[d_{ij}] = a_{ij}\tau + b_{ij}\gamma + c_{ij}\gamma^2,$$
$$[v_i] = -G\tau(1 - G^{-1}v_j\nu_j - \tau)\nu_i - G(1-\tau)\gamma\mu_i - G(\tau\nu_k + \gamma\mu_k)u_{i,k},$$
$$a_{ij} = (1 + \tau/2)\nu_i\nu_j - (u_{k,i}\nu_j + u_{k,j}\nu_i)\nu_k/2,$$
$$2b_{ij} = \mu_i\nu_j + \mu_j\nu_i - (u_{k,i}\nu_j + u_{k,j}\nu_i)\mu_k, \quad 2c_{ij} = \nu_i\nu_j. \tag{2.3}$$

In (2.3), the plus sign at the quantities calculated ahead of $\Sigma(t)$ is omitted because below we deal only with such quantities and discontinuity intensities and G is the speed of propagation of the discontinuity surfaces.

The further calculations are simplified if, following (1.3), we assume that the elastic strains e_{ij} are small although the displacement gradient tensor components $u_{i,j}$ are not small. Then, relations (1.3)–(1.5) lead to

$$e_{ij} = N_{ijkl}(d_{kl} - p_{kl}), \qquad \sigma_{ij} = R(\lambda e_{kk}\delta_{ij} + 2\mu e_{ij}),$$
$$R = \sqrt{1 - 2p_{kk} - 2(p_{kk})^2 - 2p_{km}p_{mk} - 4(p_{kk})^3/3 + 4p_{kk}p_{st}p_{ts} - 8p_{lk}p_{km}p_{ml}/3}, \tag{2.4}$$
$$N_{ijkl}K_{lkst} = \delta_{is}\delta_{jt}, \quad K_{lkst} = \delta_{lk}\delta_{st} - \delta_{lk}p_{st} - p_{lk}\delta_{st}.$$

The law of conservation of momentum implies the following dynamic conditions of compatibility of the discontinuities:

$$[\sigma_{ij}]\nu_i\nu_j = \rho(v_k\nu_k - G)[v_i]\nu_j, \qquad [\sigma_{ij}]x_{i,\beta}\nu_j = \rho(v_k\nu_k - G)[v_i]x_{i,\beta}. \tag{2.5}$$

We write (2.5) in a form that does not depend on the surface coordinates introduced. Substituting (2.2)–(2.4) into (2.5), we obtain the relations

$$(A - \rho G^2)\tau + B\gamma + D\gamma^2 = 0, \quad [(C - \rho G^2)\gamma\mu_i + (q_{ij}\tau + n_{ij}\gamma + m_{ij}\gamma^2)\nu_j + h_{ij}\gamma\mu_j]x_{i,\beta} = 0, \tag{2.6}$$

where $A = f_{ij}a_{ji} + \rho(v_k\nu_k - G)(G\tau + v_s\nu_s - Gu_{i,j}\nu_i\nu_j)$, $B = f_{ij}b_{ji} + \rho G(v_k\nu_k - G)u_{i,j}\mu_j\nu_i$, $C = \rho(2G - v_k\nu_k)v_s\nu_s - \rho G\tau(v_k\nu_k - G)$, $D = f_{ij}c_{ji}$, $q_{ij} = 2\mu RN_{ijkl}a_{lk} - \rho Gu_{i,j}(v_k\nu_k - G)$, $n_{ij} = 2\mu RN_{ijkl}b_{lk}$, $m_{ij} = 2\mu RN_{ijkl}c_{lk}$, $h_{ij} = \rho Gu_{i,j}(v_k\nu_k - G)$, and $f_{ij} = RN_{klij}(\lambda\delta_{kl} + 2\mu\nu_k\nu_l)$.

To rewrite the three equations (2.6) ($\beta = 1, 2$) so as to eliminate arbitrariness in the choice of surface coordinates, we multiply the last two equality in (2.6) by the components of the vector μ_i and the vector orthogonal to it. Ultimately, we obtain

$$(A - \rho G^2)\tau + B\gamma + D\gamma^2 = 0,$$
$$(C - \rho G^2)\gamma + (q_{ij}\tau + n_{ij}\gamma + m_{ij}\gamma^2)\mu_i\nu_j + h_{ij}\gamma\mu_j\mu_i = 0, \tag{2.7}$$
$$[(q_{ij}\tau + n_{ij}\gamma + m_{ij}\gamma^2)\nu_j + h_{ij}\gamma\mu_j]\varepsilon_{ist}\nu_s\mu_t = 0, \qquad \mu_i\mu_i = 1, \qquad \mu_i\nu_i = 0,$$

where ε_{ist} is a unit skew-symmetric tensor. It should be noted that relations (2.7) are also valid if the elastic strains are not small. In this case, the coefficients A, B, C, and D, which depend on previous strains in the medium and the surface discontinuity intensities τ and γ, include terms of higher than the first order with respect for the components of the tensor $u_{i,j}$. The same is valid for the tensors f_{ij}, h_{ij}, q_{ij}, m_{ij}, and n_{ij}, which are completely determined only by the strain ahead of the discontinuity surface. These relations are too cumbersome and are not given here.

Thus, in the presence of previous irreversible strains in a medium, even under assumption of smallness of the subsequent reversible strains, the problem reduces to the nonlinear problem of propagation of boundary perturbations in an elastoplastic medium. This is due to both the nonlinearity of representation (1.3) of the total strains d_{ij} in terms of the elastic and plastic components and the fundamental difference between the Murnaghan formula (1.4) for the presence of irreversible strains and formula (1.6) for the case of their absence.

If we assume that the previous strains, the motion ahead of $\Sigma(t)$, and the geometry of the discontinuity surfaces are known (i.e., if we set ν_j), Eqs. (2.7) represent a system of five scalar equations for the six unknowns G, τ, γ, and μ_i. Consequently, these parameters of the boundary-value problem are calculated only in the course of solution of the problem. However, if one of them is considered specified, the condition of solvability of system (2.7) gives important information on the conditions of existence of shock waves of various types.

3. One-Dimensional Plane Elastic Shock Waves. The main regularities of propagation of strong discontinuity surfaces have the clearest mechanical meaning in the case of one-dimensional plane waves [12, 13]. Therefore, we restrict ourselves to analysis of the possible solutions of (2.7) in this simplest case. We assume that irreversible strains were present in an elastoplastic half-space $x_1 \geqslant 0$ up to the time $t = 0$, and at $t = 0$, the half-space was unloaded. The accumulated plastic strains p_{11}, p_{12}, and p_{13} are considered known, and the other components of the plastic strain tensor are considered zero. The existence of nonzero irreversible strains after unloading leads to the occurrence of residual stresses and, hence, to nonzero elastic strains. Generally, the latter can also be determined by the nature of the unloading process, and, therefore, we consider them unknown.

From the time $t = 0$, let the elastoplastic half-space be shock loaded, so that the displacement vector components depend on just one spatial coordinate x_1 $[u_i = u_i(x_1, t)]$. We assume that this shock loading does not lead to new plastic flow and study the manner in which such a boundary perturbation propagates in the medium. Because strong discontinuity planes can propagate in the medium, we consider their features. In this case, system (2.7) is simplified and can be written as

$$F\tau - V[\varphi] = 0, \quad L\eta\tau + S[\varphi] = 0, \quad T\gamma(\mu_2 p_{13} - \mu_3 p_{12}) = 0, \tag{3.1}$$

where

$$\varphi = u_{2,1}^2 + u_{3,1}^2, \quad [\varphi] = 2(u_{2,1}\mu_2 + u_{3,1}\mu_3)\gamma - \gamma^2, \quad \eta = p_{12}^2 + p_{13}^2,$$

$$F = \frac{\rho_0 G^2}{1 - p_{11}} - \frac{(\lambda + 2\mu)(1 - p_{11}) - 4\mu\eta}{1 - 2p_{11} - 4\eta}, \quad V = \frac{\lambda - (\lambda - 2\mu)p_{11} + 4\mu\eta}{4(1 - p_{11})(1 - 2p_{11} - 4\eta)}, \quad L = 2\Big(\frac{\rho_0 G^2}{1 - p_{11}} - \frac{\mu}{1 - 2p_{11} - 4\eta}\Big),$$

$$T = \rho_0 G^2 - \frac{\mu}{1 - p_{11} - \eta}, \qquad S = \frac{\rho_0 G^2}{4} - \frac{\mu(1 - 2p_{11} - 2\eta)}{4(1 - p_{11})(1 - 2p_{11} - 4\eta)}.$$

The first two equations in (3.1) form a homogeneous linear equations for τ and $[\varphi]$. If τ and $[\varphi]$ vanish simultaneously, then, for $\gamma \neq 0$, the last equation in (3.1) implies that $T = 0$ or $G = \sqrt{\mu/[\rho_0(1 - p_{11} - \eta)]}$. On such a discontinuity surface, $(u_{2,1}^+)^2 + (u_{3,1}^+)^2 = (u_{2,1}^-)^2 + (u_{3,1}^-)^2$, i.e., on this surface, the intensity of the previous shear cannot change but the direction of the shear can change. In this case, μ_2 and μ_3 remain undetermined and can be calculated only taking into account the effect on the boundary. Such a discontinuity plane will be called a transverse shock wave ($\tau = 0$) or a circular polarization wave [14]. We note that the last equality in (3.1) cannot be satisfied by setting $\gamma = 0$ because for $\eta \neq 0$, this implies that $\tau = 0$.

Let the last equation in (3.1) be satisfied by virtue of the condition

$$\mu_2/\mu_3 = p_{12}/p_{13}. \tag{3.2}$$

This implies that the possible shock waves are plane polarized and their polarization planes are uniquely determined by previous irreversible strains. The speeds of propagation of such discontinuity planes are calculated by equating to zero of the determinant of the homogeneous system of the first two equations: $FS + LV\eta = 0$. From this, for G^2, we have two values:

$$G^2 = P \pm \sqrt{Q}. \tag{3.3}$$

Here

$$P = \frac{\lambda + 2\mu}{\rho_0}\,\frac{1 - 2p_{11} + p_{11}^2 - 2\eta}{2(1 - 2p_{11} - 4\eta)} + \frac{\mu}{\rho_0}\,\frac{1 - 2p_{11} - 2\eta(1 + 2p_{11}^2 + 4\eta)}{2(1 - p_{11})(1 - 2p_{11}^2 - 4\eta)},$$

$$Q = P^2 - \frac{(\lambda + 2\mu)\mu}{\rho_0^2}\,\frac{1 - p_{11}}{(1 - 2p_{11} - 4\eta)^2}.$$

The value of $G^2 = P + \sqrt{Q}$ corresponds to a quasilongitudinal shock wave. As the plastic strains tend to zero, this quantity tends to the value of $c_1^2 = (\lambda + 2\mu)\rho_0^{-1}$ equal to the squared speed of vortex-free elastic waves. However, not only does the presence of residual distortions and stresses change the speed of the given discontinuity surface but it also leads to discontinuity of the shear strains on it ($\gamma \neq 0$). This discontinuity is the greater, the higher the level of accumulated irreversible strains. If $\eta = 0$ ahead of such a discontinuity plane, the shock wave becomes longitudinal ($\gamma = 0$). In any case, the discontinuity intensities τ and γ are related to one another. If, for example, τ is known, γ is calculated from relation (3.1). By analogy with the nonlinear theory of elasticity [14, 15], such a discontinuity surface will be called a quasilongitudinal shock wave. We note that this shock wave is plane polarized in the presence of previous irreversible strains. Its polarization plane is given by relation (3.2) and does not depend on the nature of shock action on the medium.

The other plane polarized discontinuity is related to the speed of its propagation $G^2 = P - \sqrt{Q}$. The polarization plane of such a shock wave is also rigidly determined according to (3.2) by the previous irreversible strains. On this plane, both τ and γ are different from zero, and unlike in the case of a quasilongitudinal shock wave, one of the discontinuity intensities can vanish. Only if the plastic strains ahead of the given discontinuity plane tend to zero does the quantity τ also tend to zero. This discontinuity plane will be called a quasitransverse elastic shock wave [15]. As the previous plastic strains decrease, the speed of this shock wave tends to the value of $c_2 = (\mu\rho_0^{-1})^{1/2}$ equal to the speed of propagation of equivoluminal waves in the linear theory of elasticity.

Generally, for nonplanar discontinuity surfaces, the orientation of the polarization plane of a quasitransverse shock wave depends not only on previous plastic strains but also on the nature of the shock action. A circular polarization shock wave is not transverse ($\tau \neq 0$), and the components μ_i of the unit vector introduced depend on the previous strains. Unlike in the case considered above, the speed of propagation of such a discontinuity surface depends not only on previous strains but also on the discontinuity intensities γ or τ.

4. Examples of the Simplest Model Problems. Let us consider a self-similar problem of shock loading of an elastoplastic half-space in which irreversible strains were accumulated. It is assumed that the distribution of the previous plastic strains is uniform, $p_{11} = \eta_1 -$ const, $p_{12} = \eta_2 -$ const, and the remaining components of the plastic strain tensor are equal to zero. We note that this is generally valid for constant one-dimensional strains. Vanishing of p_{13} is achieved by an appropriate choice of a coordinate system. The stresses on the boundary of the half-space $x_1 = 0$ is set equal to zero (complete unloading) up to the time $t = 0$. At $x_1 > 0$, residual stresses are present. From the moment $t = 0$, let the boundary $x_1 = 0$ be subjected to constant loading, so that

$$\sigma_{11}(0,t) = \sigma_{11}^{(0)}, \quad \sigma_{21}(0,t) = \sigma_{21}^{(0)}, \quad \sigma_{31}(0,t) = \sigma_{31}^{(0)}.$$

It is known that this problem is self-similar for the variable $\chi = x_1(c_1 t)^{-1}$ and variation of parameters of the stress–strain state can occur only by discontinuities on shock waves or inside propagating simple waves. We set $\sigma_{11}^{(0)} \leqslant 0$ and $\sigma_{21}^{(0)} > 0$. As a result, for positive η_2, only shock waves can propagate over the medium. The leading edge of the perturbations propagating into the elastoplastic half-space is a quasilongitudinal shock wave propagating at speed

$$G_1 = \sqrt{P + \sqrt{Q}}$$

[in calculating P and Q according to (3.3), it is necessary to set $\eta = \eta_2^2$]. After propagation of this shock wave in the medium, the stress–strain state changes, so that

$$[\varphi] = -L\eta_2^2\tau_1/S, \qquad [u_{3,1}] = 0,$$

$$\sigma_{11} = \sigma_{11}^* + \frac{(\lambda + 2\mu)R\tau_1[(1 + L\eta_2^2/(2S))(1 - \eta_1 - 2\eta_2^2) - L\eta_2^2/(4S)]}{(1 - \eta_1)(1 - 2\eta_1 - 4\eta_2^2)}, \tag{4.1}$$

$$\sigma_{21} = \sigma_{21}^* + 2\mu R\eta_2\tau_1 \frac{1 + L\eta_2^2/(2S) - L(1 - 2\eta_1)/(8S)}{(1 - \eta_1)(1 - 2\eta_1 - 4\eta_2^2)}, \qquad \sigma_{31} = \sigma_{31}^*.$$

Here σ_{11}^*, σ_{21}^*, and σ_{31}^* are the components of the residual stress tensor, which are considered known; in the calculation of L and S according to (3.3), it is necessary to set $\eta = \eta_2^2$. The value of the longitudinal discontinuity τ_1 remains unknown and will be determined from the boundary conditions on the loaded boundary plane.

The calculations showed that in any case, a quasilongitudinal shock is followed by a quasitransverse wave:

$$G_2 = \sqrt{P - \sqrt{Q}} \geqslant \sqrt{\mu/[\rho_0(1 - \eta_1 - \eta_2^2)]} = G_3. \tag{4.2}$$

Inequality (4.2) is of fundamental significance in the formulation of boundary-value problems. After propagation of the quasitransverse shock wave, the stress state changes:

$$\sigma_{11} = \sigma_{11}^* + \frac{(\lambda+2\mu)R[U(1-\eta_1-2\eta_2^2) - L\eta_2^2\tau_1/(4S) + \eta_2\gamma_2]}{(1-\eta_1)(1-2\eta_1-4\eta_2^2)},$$

$$\sigma_{21} = \sigma_{21}^* + \frac{2\mu R[U\eta_2 + (\gamma_2 - L\eta_2\tau_1/(4S))(1-2\eta_1)/2]}{(1-\eta_1)(1-2\eta_1-4\eta_2^2)}, \quad \sigma_{31} = \sigma_{31}^*, \tag{4.3}$$

$$U = (1 + L\eta_2^2/(2S))\tau_1 + (4V/F - 2)\gamma_2\eta_2.$$

In (4.3) it is necessary to take into account that, according to (3.1), the quantities L and S are calculated for $G = G_1$, and the quantities F and V for $G = G_2$. As follows from (4.3), the stresses behind the quasitransverse discontinuity plane depend on the two independent parameters τ_1 and γ_2.

Finally, the quasitransverse shock wave is followed by a circular polarization shock wave propagating at speed $G = G_3$. Because $[\varphi] = 0$ on this discontinuity plane, it follows that the value of μ_2 is small compared to μ_3. Consequently, the changes of σ_{11} and σ_{21} on this wave are small compared to the magnitude of the stress discontinuity σ_{31}. Therefore, in the region behind the circular polarization wave, the stresses can be calculated from relations (4.3) and the relation

$$\sigma_{31} = \mu\gamma_3 \frac{(1-2\eta_1)(1-\eta_1-2\eta_2^2) - 2\eta_2^2}{(1-\eta_1)(1-\eta_1-2\eta_2^2)(1-2\eta_1-4\eta_2^2)}. \tag{4.4}$$

The parameters τ_1, γ_2, and γ_3 are obtained from the condition on the boundary of the half-space:

$$\tau_1 = \frac{(1-\eta_1)(1-2\eta_1-4\eta_2^2)}{R}\Big\{\frac{\sigma_{11}^{(0)} - \sigma_{11}^*}{\lambda+2\mu}\Big[\Big(\frac{4V}{F} - 2\Big)\eta_2^2 + \frac{1-2\eta_1}{2}\Big] - \frac{\sigma_{21}^{(0)} - \sigma_{21}^*}{2\mu}\Big[\Big(\frac{4V}{F} - 2\Big)(1-\eta_1-2\eta_2^2) + 1\Big]\eta_2\Big\},$$

$$\gamma_2 = \frac{(1-\eta_1)(1-2\eta_1-4\eta_2^2)}{R}\Big\{-\frac{\sigma_{11}^{(0)} - \sigma_{11}^*}{\lambda+2\mu}\Big(1 + \frac{L\eta_2^2}{2S} - \frac{L}{8S}(1-2\eta_1)\Big)\eta_2 \tag{4.5}$$

$$+ \frac{\sigma_{21}^{(0)} - \sigma_{21}^*}{2\mu}\Big[\Big(1 + \frac{L\eta_2^2}{2S}\Big)(1-\eta_1-2\eta_2^2) - \frac{L\eta_2^2}{4S}\Big]\Big\},$$

$$\gamma_3 = \frac{\sigma_{31}^{(0)}}{\mu}\,\frac{(1-\eta_1)(1-\eta_1-\eta_2^2)(1-2\eta_1-4\eta_2^2)}{(1-2\eta_1)(1-\eta_1-2\eta_2^2) - 2\eta_2^2}.$$

Thus, relations (4.1), (4.3), and (4.4) with τ_1, γ_2, and γ_3 calculated according to (4.5), are a solution of the problem of determining stresses under unsteady deformation of a shock-loaded half-space. In this case, it is only necessary to specify the residual stress distribution in this space, i.e., σ_{11}^* and σ_{21}^* should be known. We note again that the quantities L and S in (4.5) are calculated from relations (3.1) for $G = G_1$, and quantities F and V are calculated by the same relation for $G = G_2$. Thus, the presence of accumulated irreversible strains results in a medium leads to a change of the nature of distribution of subsequent elastic strains over the medium.

Let us consider another unsteady problem of instantaneous unloading of an elastoplastic plane layer $0 \leqslant x_1 \leqslant H$. We assume that the previous stress state in the layer corresponds to developed plastic flow [11]:

$$(\sigma_{11}^* - \sigma_{22}^*)^2 - 4\sigma_{12}^* = 4k^2. \tag{4.6}$$

The final strain state in the layer depends substantially on the history of the active irreversible deformation. For the purposes of the present work, this is necessary as the initial condition of the problem. We assume that of the components of the displacement gradient tensor only $u_{2,1}$ is not equal to zero. Moreover, we set $u_{2,1} = h -$ const. In other words, for the strain state, we adopt the condition of pure shear. In nonlinear media, this is possible only for $\sigma_{11}^*(0) \neq 0$. From (1.3) it follows that of the plastic strain tensor components, only $p_{12} = \eta_2 -$ const is nonzero. For the elastic strain tensor components in this strained state, we have

$$e_{11} = \frac{-h^2(1-2\eta_2^2) + 2\eta_2(h-2\eta_2)}{2(1-4\eta_2^2)}, \quad e_{12} = \frac{-h^2\eta_2 + h - 2\eta_2}{2(1-4\eta_2^2)}.$$

Here the constant h is given by equality (4.6) and η_2 should be specified.

At the time $t = 0$ let the constant load on the plane $x_1 = 0$ be instantaneously eliminated. By virtue of the chosen initial conditions, the problem becomes self-similar again. We introduce a new dependent variable $w(\xi)$ using the transformation $u_2 = c_2 t w(\xi)$, where $\xi = x_1(c_2 t)^{-1}$ and $c_2^2 = \mu\rho_0^{-1}$.

The equation of motion implies that

$$[(1 - 2w'\eta_2)/(1 - 4\eta_2^2) - \xi^2]w'' = 0. \tag{4.7}$$

Thus, $w' = \text{const}$ everywhere if the expression enclosed in brackets in (4.7) is not equal to zero. However, unlike in the previous case, where this expression vanished only for some values ξ, here it can vanish in the interval $[\xi^-, \xi^+]$ (simple wave). The value of ξ^+ is obtained using initial conditions, according to which $u_{2,1}^+ = w' = h$, and ξ^- is obtained using the condition of absence of elastic strains $w' = 2\eta_2$:

$$\xi^+ = \sqrt{(1 - 2\eta_2 h)/(1 - 4\eta_2^2)}, \qquad \xi^- = 1.$$

Thus, the unloading does not give rise to a shock wave. Unsteady variation of the stress–strain state results from the propagation of a simple Riemann wave.

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REFERENCES

1. G. I. Bykovtsev and L. D. Kretova, "Propagation of shock waves in elastoplastic media," *Prikl. Mat. Mekh.,* **36**, No. 1, 106–116 (1972).
2. A. A. Burenin, G. I. Bykovtsev, and V. A. Rychkov, "Velocity discontinuity surface in the dynamics of irreversible compressible media," in: *Problems of Continuum Mechanics* (to the 60th birthday of Academician V. P. Myasnikov) [in Russian], Inst. of Automatics and Control Procedures, Far-East Div., Russian Acad. of Sci. (1996), pp. 106–127.
3. V. M. Sadovskii, *Discontinuous Solutions in Problems of Dynamics of Elastoplastic Media* [in Russian], Institute of Machine Science, Moscow (1997).
4. E. N. Lee, "Elastic–plastic strain at finite strains," *Trans. ASME, J. Appl. Mech.*, **36**, No. 1, 1–6 (1969).
5. V. I. Kondaurov, "Equations of an elastoviscoplastic medium with finite deformations," *J. Appl. Mech. Tech. Phys.*, **4**, 584–590 (1982).
6. V. I. Levitas, *Large Elastoplastic Strains of Materials at High Pressure* [in Russian], Naukova Dumka, Kiev (1987).
7. A. A. Burenin, G. I. Bykovtsev, and L. V. Kovtanyuk, "On a simple model for an elastoplastic medium at finite strains," *Dokl. Ross. Akad. Nauk,* **247**, No. 2, 199–201 (1996).
8. V. P. Myasnikov, "Equations of motion of elastoplastic materials at large strains," *Vestn. Dal'nevost. Otd. Ross. Akad. Nauk*, No. 4, 8–14 (1996).
9. A. D. Chernyshev, "Constitutive equations for an elastoplastic body at finite strains," *Izv. Ross. Akad. Nauk, Mekh. Tverd. Tela*, No. 1, 120–128 (2000).
10. L. V. Kovtanyuk and M. V. Polonik, "Lamé problem of the equilibrium of a thick-wall tube made of an incompressible elastoplastic material," in: *Problems of the Continuum Mechanics and Structural Members* (to the 60th birthday of Professor G. I. Bykovtsev), Inst. of Automatics and Control Processes, Far East Div., Russian Acad. of Sci. (1998), pp. 77–96.
11. G. I. Bykovtsev and D. D. Ivlev, *Plastic Theory* [in Russian], Dal'nauka, Vladivostok (1998).
12. É. V. Lenskii, *Analytical Methods of Dynamic Nonlinear Elasticity Theory* (*Combined Nonlinearly Elastic Waves*) [in Russian], Izd. Mosk. Univ., Moscow (1983).
13. A. A. Burenin and O. V. Dudko, "Propagation of shock perturbations in a previously deformed, different-modulus elastic medium," in: *Applied Problems of the Mechanics of Deformable Solids* [in Russian], Inst. of Machine Science and Metallurgy, Far East Div., Russian Acad. of Sci., Komsomol'sk-on-Amur (1997), pp. 20–24.
14. D. R. Bland, *Nonlinear Dynamic Elasticity*, Blaisdell, Waltham (1969).
15. A. A. Burenin and A. D. Chernyshov, "Shock waves in an isotropic elastic space," *Prikl. Mat. Mekh.*, **42**, No. 4, 711–717 (1978).